Nonlocal, nonlinear, nonsmooth

Juan Pablo Borthagaray

Department of Mathematics, University of Maryland, College Park

Fractional PDEs: Theory, Algorithms and Applications
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Fractional Laplacian in $\mathbb{R}^n$

Let $s \in (0, 1)$ and $u : \mathbb{R}^n \to \mathbb{R}$ be smooth enough (belongs to Schwartz class).

- **Pseudodifferential operator:**
  \[ \mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}u(\xi). \]

- **Integral representation:**
  \[ (-\Delta)^s u(x) = C(n, s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy, \]
  where $C(n, s) = \frac{2^{2s} s \Gamma(s + \frac{n}{2})}{\pi^{n/2} \Gamma(1-s)}$ is a normalization constant.

- **Probabilistic interpretation:** related to random walks with jumps.

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  \mathcal{F} ((-\Delta)^s u) (\xi) = |\xi|^{2s} \mathcal{F} u (\xi).
  \]

- **Integral representation:**
  \[
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  \]
  where $C(n, s) = \frac{2^{2s} s \Gamma(s+\frac{n}{2})}{\pi^{n/2} \Gamma(1-s)}$ is a normalization constant.

- **Probabilistic interpretation:** related to random walks with jumps.

- **Pointwise limits as $s \to 0, 1$:**
  \[
  \lim_{s \to 0} (-\Delta)^s u = u,
  \]
  \[
  \lim_{s \to 1} (-\Delta)^s u = -\Delta u.
  \]
Integral definition for $\Omega \subset \mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let $f : \Omega \to \mathbb{R}$.

- **Boundary value problem:**

\[
\begin{cases}
(−\Delta)^s u = f & \text{in } \Omega, \\
u = 0 & \text{in } \Omega^c.
\end{cases}
\]

- **Integral representation:**

\[
(−\Delta)^s u(x) = C(n, s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) − u(y)}{|x − y|^{n+2s}} \, dy = f(x), \quad x \in \Omega.
\]

- **Boundary conditions:** imposed in $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$

\[u = 0 \quad \text{in } \Omega^c.\]

- **Probabilistic interpretation:** it is the same as over $\mathbb{R}^n$ except that particles are killed upon reaching $\Omega^c$. 

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Between the identity and the Laplacian

Solutions to fractional obstacle problems on the square $[-1, 1] \times [-1, 1]$, with $f = 0$, various $s$, and obstacle

$$\chi(x) = \max \left\{ \frac{1}{4} - \left| x - \left( -\frac{3}{4}, \frac{3}{4} \right) \right|, 0 \right\} + \max \left\{ \frac{1}{2} - \left| x - \left( \frac{1}{4}, -\frac{1}{4} \right) \right|, 0 \right\}.$$

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Some remarks

- There is not a unique way to define a “fractional Laplacian” over $\Omega$ (spectral, restricted, tempered, directional...).

- Numerical methods for the integral fractional Laplacian on bounded domains include
  - ...

- (To the best of my knowledge) these methods have been implemented mainly for linear/semilinear problems.
Goal & outline

Design **finite element methods** for nonlocal (fractional) problems. Derive **Sobolev regularity estimates** and perform a **finite element analysis** of these problems on bounded domains.

- (Linear) Dirichlet problem.
  - Regularity of solutions.
  - Finite element discretizations.
  - Reduced regularity near $\partial \Omega$: graded meshes.

- Fractional obstacle problem.

- Fractional minimal surfaces.
Function spaces

- **Fractional Sobolev spaces in $\mathbb{R}^n$:**

  $$H^s(\mathbb{R}^n) = \{ v \in L^2(\mathbb{R}^n) : |v|_{H^s(\mathbb{R}^n)} < \infty \}$$

  with

  $$\langle u, w \rangle := \frac{C(n, s)}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\,dy\,dx,$$

  $$|v|_{H^s(\mathbb{R}^n)} := \langle v, v \rangle^{\frac{1}{2}}, \quad \|v\|_{H^s(\mathbb{R}^n)} := \left( \|v\|_{L^2(\mathbb{R}^n)}^2 + |v|_{H^s(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}.$$

- **Fractional Sobolev spaces in $\Omega$:**

  $$\tilde{H}^s(\Omega) := \{ v|_\Omega : v \in H^s(\mathbb{R}^n), \supp(v) \subset \overline{\Omega} \}, \quad \|v\|_{\tilde{H}^s(\Omega)} := \|v\|_{H^s(\mathbb{R}^n)}.$$

- **Dual space:** $H^{-s}(\Omega) = \left[ \tilde{H}^s(\Omega) \right]^*$. 

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All the basic analysis tools we need have a fractional counterpart!

- Integration by parts formula
- Coercive bilinear form on a suitable space (Poincaré inequality)
- Finite elements = projection w.r.t. energy norm
- Interpolation estimates
Something old, something new...

- All the basic analysis tools we need have a fractional counterpart!
  - Integration by parts formula (see next slide)
  - Coercive bilinear form on a suitable space (Poincaré inequality) \( H^1_0(\Omega) \hookrightarrow \tilde{H}^s(\Omega) \)
  - Finite elements = projection w.r.t. energy norm
  - Interpolation estimates

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Nonlocal, nonlinear, nonsmooth
Something old, something new...

- **All the basic analysis tools we need have a fractional counterpart!**
  - Integration by parts formula ✓
  - Coercive bilinear form on a suitable space (Poincaré inequality) \( H^1_0(\Omega) \mapsto \widetilde{H}^s(\Omega) \) ✓
  - Finite elements \( = \) projection w.r.t. energy norm Galerkin orthogonality ✓
  - Interpolation estimates Lagrange interpolation \( \mapsto \) quasi-interpolation ✓

- **Nonlocality**
  - The \( H^s \)-seminorms are not additive with respect to domain partitions.
  - Functions with disjoint supports may have a non-zero inner product: if \( u, v > 0 \) on their supports, then
    \[
    \langle u, v \rangle = \frac{C(n, s)}{2} \iint_{\text{supp}(u) \times \text{supp}(v)} \frac{-2u(x)v(y)}{|x - y|^{n+2s}} \, dx \, dy < 0.
    \]
  - Singular integrals, integration on unbounded domains.
  - How smooth are solutions? Is there a lifting property?

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Integration by parts \cite{Dipierro,Ros-Oton&Valdinoci(2017)}

\[
[u, v] = \int_{\Omega} v(x)(-\Delta)^s u(x) \, dx + \int_{\Omega^c} v(x) N_s u(x) \, dx.
\]

Here,

\[
[u, v] := \frac{C(n, s)}{2} \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy,
\]

and $N_s$ is a nonlocal derivative operator,

\[
N_s u(x) := C(n, s) \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy, \quad x \in \Omega^c.
\]
Integration by parts (Dipierro, Ros-Oton & Valdinoci (2017))

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[u, v] = \int_{\Omega} v(x)(-\Delta)^s u(x) \, dx + \int_{\Omega^c} v(x) N_s u(x) \, dx.
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N_s u(x) := C(n, s) \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy, \quad x \in \Omega^c.
\]

Random walk interpretation: if the particle goes to $x \in \Omega^c$, it may return to any point $y \in \Omega$, with the probability of jumping from $x$ to $y$ being proportional to $|x - y|^{-n-2s}$.

The function $N_s u$ can be regarded as a nonlocal flux density on $\Omega^c$ into $\Omega$.  

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Dirichlet problem

Given \( f \in H^{-s}(\Omega) \), find \( u \in \tilde{H}^s(\Omega) \) such that

\[
\begin{cases}
(\Delta)^s u = f & \text{in } \Omega, \\
 u = 0 & \text{in } \Omega^c.
\end{cases}
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Dirichlet problem

Given $f \in H^{-s}(\Omega)$, find $u \in \tilde{H}^s(\Omega)$ such that

\[
\begin{align*}
(-\Delta)^s u &= f \quad \text{in } \Omega, \\
 u &= 0 \quad \text{in } \Omega^c.
\end{align*}
\]

- **Variational formulation:**
  \[
  [u, v] = (f, v) \quad \forall v \in \tilde{H}^s(\Omega),
  \]
  where $(\cdot, \cdot)$ stands for the duality pairing $H^{-s}(\Omega) \times \tilde{H}^s(\Omega)$.

- **Poincaré inequality in $\tilde{H}^s(\Omega)$:**
  \[
  \|v\|_{L^2(\Omega)} \leq c(\Omega, n, s) |v|_{H^s(\mathbb{R}^n)} \quad \forall v \in \tilde{H}^s(\Omega).
  \]
  Therefore, the form $[\cdot, \cdot] : \tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega)$ is an inner product in $\tilde{H}^s(\Omega)$, and we will write $\|v\|_{\tilde{H}^s(\Omega)} = [v, v]^{1/2}$.

- **Existence, uniqueness, and stability** follow from Lax-Milgram theorem.

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Sobolev regularity of solutions

Theorem (Vishik & Èskin (1965), Grubb (2015))

If \( f \in H^r(\Omega) \) for some \( r \geq -s \) and \( \partial \Omega \in C^\infty \), then, for all \( \varepsilon > 0 \),

\[
u \in \begin{cases} 
H^{2s+r}(\Omega) & \text{if } s + r < 1/2, \\
H^{s+1/2-\varepsilon}(\Omega) & \text{if } s + r \geq 1/2.
\end{cases}
\]

Example: if \( \Omega = B(0, r) \) and \( f \equiv 1 \), then the solution \( u \) is given by

\[
u(x) = C(r^2 - |x|^2)^{s+r},
\]

which does not belong to \( H^{s+1/2}(\Omega) \).

The regularity above is sharp!

Boundary behavior: if \( \partial \Omega \in C^\infty \) then

\[
u(x) \approx \text{dist}(x, \partial \Omega)^{s+r} v(x),
\]

with \( v \) smooth and vanishing on \( \partial \Omega \).
Sobolev regularity of solutions

Theorem (Vishik & Èskin (1965), Grubb (2015))

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Example: if \( \Omega = B(0, r) \) and \( f \equiv 1 \), then the solution \( u \) is given by

\[
u(x) = C(r^2 - |x|^2)^s_+,
\]

which does not belong to \( H^{s+1/2}(\Omega) \). The regularity above is sharp!

Boundary behavior: if \( \partial \Omega \in C^\infty \) then

\[
u(x) \approx \text{dist}(x, \partial \Omega)^s + v(x),
\]

with \( v \) smooth and vanishing on \( \partial \Omega \).
Formulation and best approximation

- **Mesh:** let $\mathcal{T}$ be a shape-regular and quasi-uniform mesh of $\Omega$ of size $h$.

- **Finite element space:** let

  $$\mathcal{V}(\mathcal{T}) = \{ v_h \in \mathcal{C}^0(\overline{\Omega}) : v_h|_K \in \mathcal{P}_1 \ \forall K \in \mathcal{T} \}.$$  

- **Discrete problem:** find $u_h \in \mathcal{V}(\mathcal{T})$ such that

  $$[u_h, v_h] = (f, v_h) \ \forall v_h \in \mathcal{V}(\mathcal{T}).$$

- **Best approximation:** since we project over $\mathcal{V}(\mathcal{T})$ with respect to the energy norm $\| \cdot \|_{\widetilde{H}^s(\Omega)}$ induced by $[\cdot, \cdot]$, we get

  $$\| u - u_h \|_{\widetilde{H}^s(\Omega)} = \min_{v_h \in \mathcal{V}(\mathcal{T})} \| u - v_h \|_{\widetilde{H}^s(\Omega)}.$$
Interpolation estimates in $\tilde{H}^{s}(\Omega)$

- **Localized estimates in $H^{s}(\Omega)$ (Faermann (2002)):**

  $$|v|^{2}_{H^{s}(\Omega)} \leq \frac{C(n, s)}{2} \sum_{K \in \mathcal{T}} \left[ \int_{K} \int_{S_{K}} \frac{|v(x) - v(y)|^{2}}{|x - y|^{n+2s}} \, dy \, dx + \frac{C(n, \sigma)}{sh_{K}^{2s}} \|v\|^{2}_{L^{2}(K)} \right],$$

  where $S_{K}$ is the patch associated with $K \in \mathcal{T}$ and $\sigma$ is the shape regularity constant of $\mathcal{T}$.

- **Quasi-interpolation (P. Ciarlet Jr (2013)):** if $\Pi_{h}$ is the Scott-Zhang operator,

  $$\int_{K} \int_{S_{K}} \frac{|(v - \Pi_{h}v)(x) - (v - \Pi_{h}v)(y)|^{2}}{|x - y|^{n+2s}} \, dy \, dx \lesssim h_{K}^{2\ell-2s} |v|^{2}_{H^{\ell}(S_{K})},$$

  where the hidden constant depends on $n$, $\sigma$, $\ell$ and blows up as $s \uparrow 1$.  

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Interpolation estimates in $\tilde{H}^s(\Omega)$

- **Localized estimates in $H^s(\Omega)$** (Faermann (2002)):
  \[
  |v|_{H^s(\Omega)}^2 \leq C(n, s) \sum_{K \in \mathcal{T}} \left( \int_K \int_{S_K} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dy \, dx + \frac{C(n, \sigma)}{s h_K^{2s}} \|v\|_{L^2(K)}^2 \right),
  \]
  where $S_K$ is the patch associated with $K \in \mathcal{T}$ and $\sigma$ is the shape regularity constant of $\mathcal{T}$.

- **Quasi-interpolation** (P. Ciarlet Jr (2013)): if $\Pi_h$ is the Scott-Zhang operator,
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  \]
  where the hidden constant depends on $n, \sigma, \ell$ and blows up as $s \uparrow 1$.

- **Error estimate for quasi-uniform meshes**:
  \[
  \|u - u_h\|_{\tilde{H}^s(\Omega)} \leq C(s, \sigma) h^{1/2} |\ln h| \|f\|_{H^{1/2-s}(\Omega)}.
  \]
Example

Take $\Omega = B(0, 1) \subset \mathbb{R}^2$ and $f = 1$. Then, the solution is given by

$$u(x) = C(1 - |x|^2)^s_+.$$

Orders of convergence in $\tilde{H}^s(\Omega)$

<table>
<thead>
<tr>
<th>$s$</th>
<th>Order (in $h$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.497</td>
</tr>
<tr>
<td>0.3</td>
<td>0.498</td>
</tr>
<tr>
<td>0.5</td>
<td>0.501</td>
</tr>
<tr>
<td>0.7</td>
<td>0.504</td>
</tr>
<tr>
<td>0.9</td>
<td>0.532</td>
</tr>
</tbody>
</table>

Discrete solution for $s = 0.5$.

Rate is quasi-optimal. Is it possible to improve the order of convergence?
Hölder regularity of solutions

**Theorem (Ros-Oton & Serra (2014))**

Let $\Omega$ be a bounded Lipschitz domain satisfying an exterior ball condition. If $f \in L^\infty(\Omega)$, then $u \in C^s(\mathbb{R}^n)$ and

$$
\|u\|_{C^s(\mathbb{R}^n)} \leq C(\Omega, s) \|f\|_{L^\infty(\Omega)}.
$$

(Recall $u(x) \approx \text{dist}(x, \partial\Omega)^s$ near $\partial\Omega$. )
Hölder regularity of solutions

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$$
\|u\|_{C^s(\mathbb{R}^n)} \leq C(\Omega, s) \|f\|_{L^\infty(\Omega)}.
$$

(Recall $u(x) \approx \text{dist}(x, \partial \Omega)^s$ near $\partial \Omega$.)

**Boundary behavior:** if $f \in C^\beta(\bar{\Omega})$ ($\beta < 2 - 2s$), then there exist constants $C_1, C_2 > 0$ such that

$$
\sup_{x, y \in \Omega} \delta(x, y)^{\beta+s} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^{\beta+2s-1}} \leq C_1, \quad \sup_{x \in \Omega} \delta(x)^{1-s} |\nabla u(x)| \leq C_2,
$$

where $\delta(x) := \text{dist}(x, \partial \Omega)$ and $\delta(x, y) = \min\{\delta(x), \delta(y)\}$. 

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Weighted fractional Sobolev regularity

Definition of space $\tilde{H}^{1+\theta}_{\alpha}(\Omega)$: let $\alpha \geq 0$ and $\theta \in (0,1)$.

$$\|v\|_{\tilde{H}^{1+\theta}_{\alpha}(\Omega)}^2 := \|v\|_{\tilde{H}^{1}_{\alpha}(\Omega)}^2 + \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|\nabla v(x) - \nabla v(y)|^2}{|x - y|^{n+2\theta}} \delta(x, y)^{2\alpha} \, dx \, dy,$$

with $\|v\|_{\tilde{H}^{1}_{\alpha}(\Omega)} = \|(v + \nabla v) \, \delta(\cdot)\alpha\|_{L^2(\Omega)}$. 

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Nonlocal, nonlinear, nonsmooth
Weighted fractional Sobolev regularity

**Definition of space** \( \widetilde{H}_{\alpha}^{1+\theta}(\Omega) \): let \( \alpha \geq 0 \) and \( \theta \in (0, 1) \).

\[
\|v\|_{\widetilde{H}_{\alpha}^{1+\theta}(\Omega)}^2 := \|v\|_{H_{\alpha}^1(\Omega)}^2 + \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|\nabla v(x) - \nabla v(y)|^2}{|x - y|^{n+2\theta}} \delta(x, y)^{2\alpha} \, dx \, dy,
\]

with \( \|v\|_{H_{\alpha}^1(\Omega)} = \|(v + \nabla v) \delta(\cdot)\alpha\|_{L^2(\Omega)} \).

---

**Theorem (Acosta & B. (2017))**

Let \( \Omega \) be a bounded Lipschitz domain satisfying an exterior ball condition, \( f \in C^{1-s}(\overline{\Omega}) \), and \( \varepsilon > 0 \) be small. Then, the solution \( u \) of the linear Dirichlet problem belongs to \( \widetilde{H}_{1/2-\varepsilon}^{1+s-2\varepsilon}(\Omega) \) and satisfies the estimate

\[
\|u\|_{\widetilde{H}_{1/2-\varepsilon}^{1+s-2\varepsilon}(\Omega)} \leq \frac{C(\Omega, s)}{\varepsilon} \|f\|_{C^{1-s}(\overline{\Omega})}.
\]
Error estimates in graded meshes

- **Weighted fractional Poincaré inequality:** if $S$ is star-shaped with respect to a ball, $d_S$ is the diameter of $S$, and $\overline{v} = \int_S v$, then

$$
\|v - \overline{v}\|_{L^2(S)} \lesssim d_S^{s-\alpha} |v|_{H^s_\alpha(S)}.
$$

- **Weighted quasi-interpolation:** for the SZ quasi-interpolation operator $\Pi_h$,

$$
\int_{K} \int_{S_K} \frac{|(v - \Pi_h v)(x) - (v - \Pi_h v)(y)|^2}{|x - y|^{n+2s}} dydx \lesssim h_K^{1-2\varepsilon} |v|_{H^{1+s-2\varepsilon}_{\frac{1}{2}-\varepsilon}(S_K)}^2.
$$
Error estimates in graded meshes

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$$\int_K \int_{S_K} \frac{|(v - \Pi_h v)(x) - (v - \Pi_h v)(y)|^2}{|x - y|^{n+2s}} dy dx \lesssim h_K^{1-2\varepsilon} |v|_{H^{1+s-2\varepsilon}(S_K)}^2.$$

**Energy error estimate** (Acosta & B. (2017)): let $n = 2$ and $\mathcal{T}$ be a graded mesh satisfying

$$h_K \leq C(\sigma) \begin{cases} h^2, & K \cap \partial \Omega \neq \emptyset, \\ h \operatorname{dist}(K, \partial \Omega)^{1/2}, & K \cap \partial \Omega = \emptyset, \end{cases}$$

whence $\# \mathcal{T} \approx h^{-2} |\log h|$. Then,

$$\|u - u_h\|_{H^s(\Omega)} \lesssim h |\log h| \|f\|_{C^{1-s}(\overline{\Omega})}.$$
Numerical experiment

**Exact solution:** if $\Omega = B(0, 1) \subset \mathbb{R}^2$ and $f = 1$, then $u(x) = C(r^2 - |x|^2)^s$.

<table>
<thead>
<tr>
<th>Value of $s$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
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<tbody>
<tr>
<td>Uniform $T$</td>
<td>0.497</td>
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<td>0.500</td>
<td>0.501</td>
<td>0.505</td>
<td>0.504</td>
<td>0.503</td>
<td>0.532</td>
</tr>
<tr>
<td>Graded $T$</td>
<td>1.066</td>
<td>1.040</td>
<td>1.019</td>
<td>1.002</td>
<td>1.066</td>
<td>1.051</td>
<td>0.990</td>
<td>0.985</td>
<td>0.977</td>
</tr>
</tbody>
</table>
Obstacle problem (with R. Nochetto & A. Salgado)

Given two smooth enough functions $f, \chi : \Omega \to \mathbb{R}$, find $u : \mathbb{R}^n \to \mathbb{R}$, supported in $\Omega$, such that

$$
\begin{align*}
    u &\geq \chi \quad \text{in } \Omega, \\
    (-\Delta)^s u &\geq f \quad \text{in } \Omega, \\
    (-\Delta)^s u &= f \quad \text{whenever } u > \chi.
\end{align*}
$$
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Given two smooth enough functions \( f, \chi : \Omega \to \mathbb{R} \), find \( u : \mathbb{R}^n \to \mathbb{R} \), supported in \( \Omega \), such that

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\end{align*}
\]

Can equivalently be written as a variational inequality:

Find \( u \in \mathcal{K} \) such that

\[
[u, u - v] \leq (f, u - v) \quad \forall v \in \mathcal{K},
\]

where \( \mathcal{K} \) denotes the convex set \( \mathcal{K} = \{ v \in \tilde{H}^s(\Omega) : v \geq \chi \text{ a.e. in } \Omega \} \).
Assumptions

- **Domain:** $\partial \Omega$ is Lipschitz, and satisfies an exterior ball condition.

- **Data:** from now on,

  $$\chi \in C^{2,1}(\Omega), \quad 0 \leq f \in \mathcal{F}_s(\Omega) = \begin{cases} C^{2,1-2s}(\Omega), & s \in (0, \frac{1}{2}) \\ C^{1,2-2s}(\Omega), & s \in \left[\frac{1}{2}, 1\right) \end{cases}.$$  

- We assume that $\chi < 0$ on $\partial \Omega$, so that
  
  - the behavior of solutions near $\partial \Omega$ is dictated by an elliptic (linear) problem;
  
  - the nonlinearity is constrained to the interior of the domain.
Assumptions

- **Domain:** $\partial \Omega$ is Lipschitz, and satisfies an exterior ball condition.

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  - the behavior of solutions near $\partial \Omega$ is dictated by an elliptic (linear) problem;
  
  - the nonlinearity is constrained to the interior of the domain.

- **Non-locality:** gluing interior and boundary estimates is not straightforward!
  
  If $\eta \equiv 1$ in a neighborhood of $x_0$, then it does not follow that

$$(-\Delta)^s(\eta u)(x_0) = (-\Delta)^s u(x_0).$$
Theorem (Caffarelli, Salsa & Silvestre (2008))

For the obstacle problem in \( \mathbb{R}^n \), if \( f \in \mathcal{F}_s(\mathbb{R}^n) \) and \( \chi \in C^{2,1}(\mathbb{R}^n) \), then the solution \( u \) belongs to \( C^{1,s}(\mathbb{R}^n) \).

(In particular, \( u \in H^{1+s-\varepsilon}_{\text{loc}}(\mathbb{R}^n) \) for all \( \varepsilon > 0 \).)
Regularity in $\mathbb{R}^n$

Theorem (Caffarelli, Salsa & Silvestre (2008))

For the obstacle problem in $\mathbb{R}^n$, if $f \in \mathcal{F}_s(\mathbb{R}^n)$ and $\chi \in C^{2,1}(\mathbb{R}^n)$, then the solution $u$ belongs to $C^{1,s}(\mathbb{R}^n)$.

(In particular, $u \in H^{1+s-\varepsilon}_{loc}(\mathbb{R}^n)$ for all $\varepsilon > 0$.)

Moral: free boundary regularity is not any worse than boundary regularity for the linear problem.

Hope: prove regularity in weighted Sobolev spaces.
Regularity for the obstacle problem on $\Omega$

- **Interior regularity**: Caffarelli-Salsa-Silvestre’s theorem + localization argument.

- **Boundary regularity**: use the result for the linear Dirichlet problem.

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Regularity for the obstacle problem on $\Omega$

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**Theorem**

Let $u \in \tilde{H}^s(\Omega)$ be the solution to the fractional obstacle problem. Then, for every $\varepsilon > 0$ we have that $u \in \tilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)$ with the estimate

$$\|u\|_{\tilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)} \leq \frac{C}{\varepsilon},$$

with $C > 0$ depending on $\chi, s, n, \Omega, \|f\|_{\mathcal{F}_s(\Omega)}$. 
Finite element approximation

- **Discrete problem:** find $u_h \in K_h = \{ v_h \in V_h : v_h \geq \Pi_h \varphi \}$ such that

  $$\left[ u_h, u_h - v_h \right] \leq (f, u_h - v_h) \quad \forall v_h \in K_h.$$ 

- Weighted Sobolev regularity $\Rightarrow$ **graded meshes.**
Finite element approximation

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  \[
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  \]

- **Weighted Sobolev regularity** \( \Rightarrow \) **graded meshes.**

- **Error bound:** writing
  \[
  \| u - u_h \|_{H^s(\Omega)}^2 = \left[ u - u_h, u - \Pi_h u \right] + \left[ u - u_h, \Pi_h u - u_h \right],
  \]
  we reach
  \[
  \frac{1}{2} \| u - u_h \|_{H^s(\Omega)}^2 \leq \frac{1}{2} \| u - \Pi_h u \|_{H^s(\Omega)}^2 + \left[ u - u_h, \Pi_h u - u_h \right].
  \]

- **Interpolation error** can be bounded by
  \[
  \| u - \Pi_h u \|_{H^s(\Omega)} \leq C h^{1-2\varepsilon} \| u \|_{\tilde{H}^{1+s-2\varepsilon}(\Omega)}.\]
Thus,

\[ \| u - u_h \|_{H^s(\Omega)}^2 \leq Ch^{2(1-2\varepsilon)} \| u \|_{H^{1+s-2\varepsilon}(\Omega)}^2 + (u - u_h, \Pi_h u - u_h)_s. \]
Thus,
\[ \|u - u_h\|_{H^s(\Omega)}^2 \leq Ch^{2(1-2\epsilon)} \|u\|_{H_{1/2-\epsilon}^{1+s}(\Omega)}^2 + (u - u_h, \Pi_h u - u_h)_s. \]

- **Second term in RHS:** integrate by parts and use discrete variational inequality,
\[ (u - u_h, \Pi_h u - u_h)_s \leq \sum_{T \in \mathcal{T}} \int_T (\Pi_h (u - \chi) - (u - \chi)) ((-\Delta)^s u - f). \]
Thus,
\[ \| u - u_h \|_{H^s(\Omega)}^2 \leq C h^{2(1-2\varepsilon)} \| u \|_{H^{1+s-2\varepsilon}(\Omega)}^2 + (u - u_h, \Pi_h u - u_h)_s. \]

- **Second term in RHS:** integrate by parts and use discrete variational inequality,
\[ (u - u_h, \Pi_h u - u_h)_s \leq \sum_{T \in T} \int_T (\Pi_h (u - \chi) - (u - \chi)) ((-\Delta)^s u - f). \]

Using the interior regularity \( u \in C^{1,s}(\Omega) \)
we deduce:
- \((-\Delta)^s u \in C^{1-s}(\Omega),\)
- \(u - \chi \in C^{1,s}(\Omega).\)

So, in these elements we have
\[ |((-\Delta)^s u - f) (\Pi_h (u - \chi) - (u - \chi))| \leq C h^2. \]
Theorem

$0 \leq f \in \mathcal{F}_s(\overline{\Omega})$ and assume that $\chi \in \mathcal{C}^{2,1}(\Omega)$ is such that $\chi < 0$ on $\partial \Omega$. Considering shape-regular graded meshes as before, if $h$ is sufficiently small, then it holds that

$$\| u - u_h \|_{\tilde{H}^s(\Omega)} \lesssim h |\log h|.$$
Numerical experiments

**Problem:** let $\Omega = B(0, 1) \subset \mathbb{R}^2$, and consider $f, \chi$ so that the exact solution is

$$u(x) = (1 - |x|^2)^s + p_2^s(x),$$

where $p_2^s$ is a certain Jacobi polynomial of degree two.

Left: $s = 0.1$; right: $s = 0.9$. The rate observed in both cases is $\approx h$. 
Qualitative behavior

**Problem:** let $\Omega = B(0, 1) \subset \mathbb{R}^2$, $f = 0$ and

$$\chi(x) = \frac{1}{2} - |x - x_0|, \text{ with } x_0 = (1/4, 1/4).$$
Fractional minimal surfaces  (preliminary work with R. Nochetto & W. Li)

- **Interaction:** given $s \in (0, 1/2)$ and two disjoint sets $A, B \subset \mathbb{R}^n$, define
  \[
  I(A, B) := \int_A \int_B \frac{1}{|x - y|^{n+2s}} \, dy \, dx.
  \]

- **Problem:** suppose we are given $\Omega, \tilde{E} \subset \mathbb{R}^n$ with $\tilde{E} \cap \Omega = \emptyset$. We want to define an extension $E$ of $\tilde{E}$ into $\Omega$ so that it minimizes a certain nonlocal perimeter.
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Minimize $I(E, E^c)$ among all extensions $E$: take care of interactions

- between $E \cap \Omega$ and $\mathbb{R}^n \setminus E$,
- between $\tilde{E}$ and $\Omega \setminus E$. 

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Nonlocal $s$-perimeter of $E$ in $\Omega$: (Caffarelli, Roquejoffre & Savin (2010))

$$\text{Per}_s(E, \Omega) := I(E \cap \Omega, \mathbb{R}^n \setminus E) + I(E \setminus \Omega, \Omega \setminus E).$$

Minimal sets: a measurable set $E \subset \mathbb{R}^n$ is $s$-minimal in $\Omega$ if, for every measurable set $F$ such that $E \setminus \Omega = F \setminus \Omega$,

$$\text{Per}_s(E, \Omega) \leq \text{Per}_s(F, \Omega).$$

Euler-Lagrange equation: a set $E$ is $s$-minimal in $\Omega$ if and only if

$$(-\Delta)^s (\chi_E - \chi_{\mathbb{R}^n \setminus E}) = 0, \text{ along } \partial E.$$
Graph minimal surfaces

Assume $\Omega = \Omega_0 \times \mathbb{R}$, and that

$$\tilde{E} = \{x = (x', x_n) \in \mathbb{R}^n : x_n \leq u_0(x')\},$$

where $u_0 : \mathbb{R}^{n-1} \setminus \Omega_0 \rightarrow \mathbb{R}$ is given.
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where $u_0 : \mathbb{R}^{n-1} \setminus \Omega_0 \to \mathbb{R}$ is given.

We seek for $u : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $u = u_0$ in $\mathbb{R}^n \setminus \Omega_0$, and

$$\int_{\mathbb{R}^{n-1}} g_s \left( \frac{u(y') - u(x')}{|x' - y'|} \right) \frac{u(y') - u(x')}{|x' - y'|^{n-1+2(s+1/2)}} dy' = 0 \text{ in } \Omega_0,$$

where

$$g_s(r) = \frac{1}{r} \int_0^r \frac{1}{(1 + \rho^2)^{n+2s}} d\rho.$$

Finding an $s$-nonlocal minimal surface in $\mathbb{R}^n$ becomes a nonhomogeneous problem for a nonlinear, degenerate diffusion operator of order $s + \frac{1}{2}$ in $\mathbb{R}^{n-1}$.
Discretization

- **Finite element space**: let
  \[ \mathbb{V}(\mathcal{T}) = \{ v_h \in C^0(\Omega) : v_h|_K \in P_1 \forall K \in \mathcal{T} \}. \]

- **Discrete problem**: find \( u_h \in \mathbb{V}(\mathcal{T}) \) such that \( u_h = \Pi_h u_0 \) in \( \mathbb{R}^{n-1} \setminus \Omega_0 \) and, for all \( v_h \in \mathbb{V}(\mathcal{T}) \),
  \[ \iint \mathcal{g}_s \left( \frac{u_h(y') - u_h(x')}{|x' - y'|} \right) \frac{(u_h(y') - u_h(x'))(v_h(y') - v_h(x'))}{|x' - y'|^{n+2s}} dy' = 0. \]

- **\( L^2 \)-gradient flow**: initial guess \( u_h^0 \in \mathbb{V}(\mathcal{T}) \) and time step \( \tau \). Given \( u_h^k \in \mathbb{V}(\mathcal{T}) \), find \( u_h^{k+1} \in \mathbb{V}(\mathcal{T}) \) such that
  \[ \frac{1}{\tau} \left( u_h^{k+1} - u_h^k, \varphi_i \right) = \iint \mathcal{g}_s \left( \frac{u_h^k(y') - u_h^k(x')}{|x' - y'|} \right) \frac{(u_h^k(y') - u_h^k(x'))(\varphi_i(y') - \varphi_i(x'))}{|x' - y'|^{n+2s}} dy', \]
  \[ \forall 1 \leq i \leq \mathcal{N}. \]
Energy

The solution $u$ minimizes the energy

$$I_s[u] = \iint_{(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \setminus (\Omega_0^c \times \Omega_0^c)} G_s \left( \frac{u(x) - u(y)}{|x - y|} \right) \frac{1}{|x - y|^{n-2+2s}} \, dy \, dx,$$

where $G_s$ is defined as

$$G_s(a) := \int_0^a \frac{a - \rho}{(1 + \rho^2)^{\frac{n+2s}{2}}} \, d\rho \quad (G'_s = g_s).$$

Since $a \leq C(G_s(a) + 1)$, we have

$$|u|_{W^{1,2s}(\Omega_0)} \leq C I_s[u] + C(\Omega_0).$$
Convergence

- **Open question:** how regular are nonlocal minimal surfaces?

- **Stickiness phenomenon:** boundary datum may not be attained continuously!

  (Dipierro, Savin & Valdinoci (2017))
**Convergence**

- **Open question:** how regular are nonlocal minimal surfaces?

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**Theorem (energy consistency)**

If $u \in W^{2t}_{1}(\Omega_0)$ for some $t > s$, then

$$\lim_{h \to 0} I_s[u_h] = I_s[u].$$

**Theorem (convergence)**

If we have energy consistency, then

$$\lim_{h \to 0} \|u - u_h\|_{W^{2s'}_{1}(\Omega_0)} = 0, \quad \forall s' \in [0, s).$$
Experiments

Problem: $\Omega = B(0, 1), u_0 = \chi_{B(0,3/2)}$ and $s = 0.25$. 
Experiments

**Problem:** $\Omega = B(0, 1) \setminus B(0, 1/2)$, $u_0 = \chi_{B(0,1/2)}$ and $s = 0.25$. 

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Concluding remarks

- **Fractional Laplacian** \((-\Delta)^s\): nonlocal operator of order \(0 < 2s < 2\). Computational challenges include dealing with non-integrable singularities and unbounded domains.

- **Boundary behavior**: solutions of the problems discussed behave as \(\text{dist}(x, \partial \Omega)^s\) ⇒ characterize regularity in weighted Sobolev spaces ⇒ use graded meshes.

- **Fractional obstacle problem**: behavior near the free boundary may not be any worse than behavior near \(\partial \Omega\).

- **Minimal surfaces**: leads to nonlinear, degenerate diffusion problem. Solutions may exhibit discontinuities near \(\partial \Omega\).
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Thank you!