Nonlocal, nonlinear, nonsmooth

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Fractional Laplacian in $\mathbb{R}^n$

Let $s \in (0,1)$ and $u : \mathbb{R}^n \to \mathbb{R}$ be smooth enough (belongs to Schwartz class).

- **Pseudodifferential operator:**

$$\mathcal{F}\left((-\Delta)^s u\right)(\xi) = |\xi|^{2s} \mathcal{F}u(\xi).$$

- **Integral representation:**

$$(-\Delta)^s u(x) = C(n,s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy,$$

  where $C(n,s) = \frac{2^{2s} s \Gamma(s+\frac{n}{2})}{\pi^{n/2}\Gamma(1-s)}$ is a normalization constant.

- **Probabilistic interpretation:** related to random walks with jumps.

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- **Probabilistic interpretation:** related to random walks with jumps.
- **Pointwise limits as $s \to 0, 1$:**

$$\lim_{s\to 0} (-\Delta)^s u = u,$$
$$\lim_{s\to 1} (-\Delta)^s u = -\Delta u.$$

Integral definition for $\Omega \subset \mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let $f : \Omega \to \mathbb{R}$.

- **Boundary value problem:**

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

- **Integral representation:**

$$(-\Delta)^s u(x) = C(n,s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy = f(x), \quad x \in \Omega.$$

- **Boundary conditions:** imposed in $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$

$$u = 0 \quad \text{in } \Omega^c.$$

- **Probabilistic interpretation:** it is the same as over $\mathbb{R}^n$ except that particles are killed upon reaching Ω^c.

Between the identity and the Laplacian

Solutions to fractional obstacle problems on the square $[-1,1] \times [-1,1]$, with $f = 0$, various s, and obstacle

$$\chi(x) = \max\left\{\frac{1}{4} - \left|x - \left(-\frac{3}{4}, \frac{3}{4}\right)\right|, 0\right\} + \max\left\{\frac{1}{2} - \left|x - \left(\frac{1}{4}, -\frac{1}{4}\right)\right|, 0\right\}.$$

Some remarks

- There is not a unique way to define a "fractional Laplacian" over Ω (spectral, restricted, tempered, directional...).
- Numerical methods for the integral fractional Laplacian on bounded domains include
  - Finite elements (on integral representation): D'Elia & Gunzburger (2013), Ainsworth & Glusa (2018).
  - Finite differences: Huang & Oberman (2014), Duo, van Wyk & Zhang (2018).
  - Walk-on-spheres method: Kyprianou, Osojnik & Shardlow (2017).
  - Collocation methods: Zeng, Zhang & Karniadakis (2015), Acosta, B., Bruno & Maas (2018)
  - Finite elements (using Dunford-Taylor representation): Bonito, Lei & Pasciak (2017).
  - ...
- (To the best of my knowledge) these methods have been implemented mainly for linear/semilinear problems.

Goal & outline

Design **finite element methods** for nonlocal (fractional) problems. Derive **Sobolev regularity estimates** and perform a **finite element analysis** of these problems on bounded domains.

- (Linear) Dirichlet problem.
  - Regularity of solutions.
  - Finite element discretizations.
  - Reduced regularity near $\partial\Omega$: graded meshes.
- Fractional obstacle problem.
- Fractional minimal surfaces.

Function spaces

- **Fractional Sobolev spaces in $\mathbb{R}^n$:**

$$H^s(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) \colon |v|_{H^s(\mathbb{R}^n)} < \infty \right\}$$

with

$$\langle u, w \rangle := \frac{C(n,s)}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dydx,$$

$$|v|_{H^s(\mathbb{R}^n)} := \langle v, v \rangle^{\frac{1}{2}}, \quad \|v\|_{H^s(\mathbb{R}^n)} := \left(\|v\|^2_{L^2(\mathbb{R}^n)} + |v|^2_{H^s(\mathbb{R}^n)} \right)^{\frac{1}{2}}.$$

- **Fractional Sobolev spaces in Ω:**

$$\widetilde{H}^s(\Omega) := \left\{ v|_\Omega : v \in H^s(\mathbb{R}^n), \ \mathrm{supp}(v) \subset \overline{\Omega} \right\}, \quad \|v\|_{\widetilde{H}^s(\Omega)} := \|v\|_{H^s(\mathbb{R}^n)}.$$

- **Dual space:** $H^{-s}(\Omega) = \left[\widetilde{H}^s(\Omega) \right]^*$.

Something old, something new...

- **All the basic analysis tools we need have a fractional counterpart!**
  - Integration by parts formula
  - Coercive bilinear form on a suitable space (Poincaré inequality)
  - Finite elements $=$ projection w.r.t. energy norm
  - Interpolation estimates

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  - Integration by parts formula (see next slide) ✓
  - Coercive bilinear form on a suitable space (Poincaré inequality) $H_0^1(\Omega) \mapsto \widetilde{H}^s(\Omega)$ ✓
  - Finite elements $=$ projection w.r.t. energy norm Galerkin orthogonality ✓
  - Interpolation estimates Lagrange interpolation $\mapsto$ quasi-interpolation ✓

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- **Nonlocality**
  - The H^s-seminorms are not additive with respect to domain partitions.
  - Functions with disjoint supports may have a non-zero inner product: if $u, v > 0$ on their supports, then

$$\langle u, v \rangle = \frac{C(n,s)}{2} \iint_{\mathrm{supp}(u) \times \mathrm{supp}(v)} \frac{-2\, u(x)\, v(y)}{|x-y|^{n+2s}} \, dx \, dy < 0.$$

  - Singular integrals, integration on unbounded domains.
  - How smooth are solutions? Is there a lifting property?

Integration by parts (Dipierro, Ros-Oton & Valdinoci (2017))

$$[\![u,v]\!] = \int_\Omega v(x)(-\Delta)^s u(x)\,dx + \int_{\Omega^c} v(x)\,\mathcal{N}_s u(x)\,dx.$$

Here,

$$[\![u,v]\!] := \frac{C(n,s)}{2} \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy,$$

and $\mathcal{N}_s$ is a nonlocal derivative operator,

$$\mathcal{N}_s u(x) := C(n,s) \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \quad x \in \Omega^c.$$

Integration by parts (Dipierro, Ros-Oton & Valdinoci (2017))

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Random walk interpretation: if the particle goes to $x \in \Omega^c$, it may return to any point $y \in \Omega$, with the probability of jumping from x to y being proportional to $|x-y|^{-n-2s}$.

The function $\mathcal{N}_s u$ can be regarded as a nonlocal flux density on Ω^c into Ω.

Dirichlet problem

Given $f \in H^{-s}(\Omega)$, find $u \in \widetilde{H}^s(\Omega)$ such that

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

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- **Variational formulation:**

$$[\![u, v]\!] = (f, v) \quad \forall v \in \widetilde{H}^s(\Omega),$$

  where $(\cdot, \cdot)$ stands for the duality pairing $H^{-s}(\Omega) \times \widetilde{H}^s(\Omega)$.

- **Poincaré inequality in $\widetilde{H}^s(\Omega)$:**

$$\|v\|_{L^2(\Omega)} \leq c(\Omega, n, s)|v|_{H^s(\mathbb{R}^n)} \quad \forall v \in \widetilde{H}^s(\Omega).$$

  Therefore, the form $[\![\cdot, \cdot]\!] : \widetilde{H}^s(\Omega) \times \widetilde{H}^s(\Omega)$ is an inner product in $\widetilde{H}^s(\Omega)$, and we will write $\|v\|_{\widetilde{H}^s(\Omega)} = [\![v, v]\!]^{1/2}$.

- Existence, uniqueness, and stability follow from Lax-Milgram theorem.

Sobolev regularity of solutions

Theorem (Vishik & Èskin (1965), Grubb (2015))

If $f \in H^r(\Omega)$ *for some* $r \geq -s$ *and* $\partial\Omega \in C^\infty$, *then, for all* $\varepsilon > 0$,

$$u \in \begin{cases} H^{2s+r}(\Omega) & \text{if } s + r < 1/2, \\ H^{s+1/2-\varepsilon}(\Omega) & \text{if } s + r \geq 1/2. \end{cases}$$

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- **Example:** if $\Omega = B(0, r)$ and $f \equiv 1$, then the solution u is given by

$$u(x) = C(r^2 - |x|^2)_+^s,$$

  which does not belong to $H^{s+1/2}(\Omega)$. **The regularity above is sharp!**

- **Boundary behavior:** if $\partial\Omega \in C^\infty$ then

$$u(x) \approx \mathrm{dist}(x, \partial\Omega)^s + v(x),$$

  with v smooth and vanishing on $\partial\Omega$.

Formulation and best approximation

- **Mesh:** let $\mathcal{T}$ be a shape-regular and quasi-uniform mesh of Ω of size h.

- **Finite element space:** let

$$\mathbb{V}(\mathcal{T}) = \{v_h \in C^0(\overline{\Omega}) \colon v_h\big|_K \in \mathcal{P}_1 \ \forall K \in \mathcal{T}\}.$$

- **Discrete problem:** find $u_h \in \mathbb{V}(\mathcal{T})$ such that

$$[\![u_h, v_h]\!] = (f, v_h) \quad \forall\, v_h \in \mathbb{V}(\mathcal{T}).$$

- **Best approximation:** since we project over $\mathbb{V}(\mathcal{T})$ with respect to the energy norm $\|\cdot\|_{\widetilde{H}^s(\Omega)}$ induced by $[\![\cdot,\cdot]\!]$, we get

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} = \min_{v_h \in \mathbb{V}(\mathcal{T})} \|u - v_h\|_{\widetilde{H}^s(\Omega)}.$$

Interpolation estimates in $\widetilde{H}^s(\Omega)$

- **Localized estimates in** $H^s(\Omega)$ (Faermann (2002)):

$$|v|^2_{H^s(\Omega)} \le \frac{C(n,s)}{2} \sum_{K\in\mathcal{T}} \left[\int_K \int_{S_K} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dydx + \frac{C(n,\sigma)}{sh_K^{2s}} \|v\|^2_{L^2(K)} \right],$$

  where S_K is the patch associated with $K \in \mathcal{T}$ and σ is the shape regularity constant of $\mathcal{T}$.

- **Quasi-interpolation** (P. Ciarlet Jr (2013)): if Π_h is the Scott-Zhang operator,

$$\int_K \int_{S_K} \frac{|(v-\Pi_h v)(x) - (v-\Pi_h v)(y)|^2}{|x-y|^{n+2s}}\, dy\, dx \lesssim h_K^{2\ell-2s} |v|^2_{H^\ell(S_K)},$$

  where the hidden constant depends on n, σ, ℓ and blows up as $s \uparrow 1$.

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  where the hidden constant depends on n, σ, ℓ and blows up as $s \uparrow 1$.

- **Error estimate for quasi-uniform meshes:**

$$\|u-u_h\|_{\widetilde{H}^s(\Omega)} \leq C(s,\sigma) h^{\frac{1}{2}} |\ln h|\, \|f\|_{H^{1/2-s}(\Omega)}.$$

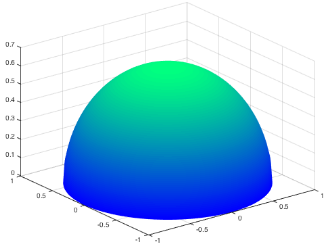
Example

Take $\Omega = B(0,1) \subset \mathbb{R}^2$ and $f = 1$. Then, the solution is given by

$$u(x) = C(1 - |x|^2)_+^s.$$

Orders of convergence in $\widetilde{H}^s(\Omega)$

s	Order (in h)
0.1	0.497
0.3	0.498
0.5	0.501
0.7	0.504
0.9	0.532



Discrete solution for $s = 0.5$.

Rate is quasi-optimal. Is it possible to improve the order of convergence?

Hölder regularity of solutions

Theorem (Ros-Oton & Serra (2014))

Let Ω be a bounded Lipschitz domain satisfying an exterior ball condition. If $f \in L^\infty(\Omega)$, then $u \in C^s(\mathbb{R}^n)$ and

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C(\Omega, s)\|f\|_{L^\infty(\Omega)}.$$

(Recall $u(x) \approx \mathrm{dist}(x, \partial\Omega)^s$ near $\partial\Omega$.)

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(Recall $u(x) \approx \mathrm{dist}(x, \partial\Omega)^s$ near $\partial\Omega$.)

- **Boundary behavior:** if $f \in C^\beta(\overline{\Omega})$ ($\beta < 2 - 2s$), then there exist constants $C_1, C_2 > 0$ such that

$$\sup_{x,y\in\Omega} \delta(x,y)^{\beta+s} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\beta+2s-1}} \leq C_1, \qquad \sup_{x\in\Omega} \delta(x)^{1-s}|\nabla u(x)| \leq C_2,$$

where $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ and $\delta(x,y) = \min\{\delta(x), \delta(y)\}$.

Weighted fractional Sobolev regularity

- **Definition of space $\widetilde{H}^{1+\theta}_\alpha(\Omega)$:** let $\alpha \geq 0$ and $\theta \in (0,1)$.

$$\|v\|^2_{\widetilde{H}^{1+\theta}_\alpha(\Omega)} := \|v\|^2_{H^1_\alpha(\Omega)} + \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|\nabla v(x) - \nabla v(y)|^2}{|x-y|^{n+2\theta}}\, \delta(x,y)^{2\alpha} dx\, dy,$$

with $\|v\|_{H^1_\alpha(\Omega)} = \|(v + \nabla v)\,\delta(\cdot)^\alpha\|_{L^2(\Omega)}$.

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with $\|v\|_{H^1_{\alpha}(\Omega)} = \|(v + \nabla v)\, \delta(\cdot)^{\alpha}\|_{L^2(\Omega)}$.

Theorem (Acosta & B. (2017))

Let Ω be a bounded Lipschitz domain satisfying an exterior ball condition, $f \in C^{1-s}(\overline{\Omega})$, and $\varepsilon > 0$ be small. Then, the solution u of the linear Dirichlet problem belongs to $\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)$ and satisfies the estimate

$$\|u\|_{\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)} \leq \frac{C(\Omega, s)}{\varepsilon} \|f\|_{C^{1-s}(\overline{\Omega})}.$$

Error estimates in graded meshes

- **Weighted fractional Poincaré inequality:** if S is star-shaped with respect to a ball, d_S is the diameter of S, and $\overline{v} = \fint_S v$, then
$$\|v - \overline{v}\|_{L^2(S)} \lesssim d_S^{s-\alpha} |v|_{H^s_\alpha(S)}.$$

- **Weighted quasi-interpolation:** for the SZ quasi-interpolation operator Π_h,
$$\int_K \int_{S_K} \frac{|(v-\Pi_h v)(x) - (v-\Pi_h v)(y)|^2}{|x-y|^{n+2s}} dydx \lesssim h_K^{1-2\varepsilon} |v|^2_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(S_K)}.$$

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Energy error estimate (Acosta & B. (2017)): let $n = 2$ and $\mathcal{T}$ be a graded mesh satisfying

$$h_K \leq C(\sigma) \begin{cases} h^2, & K \cap \partial\Omega \neq \emptyset, \\ h \operatorname{dist}(K, \partial\Omega)^{1/2}, & K \cap \partial\Omega = \emptyset, \end{cases}$$

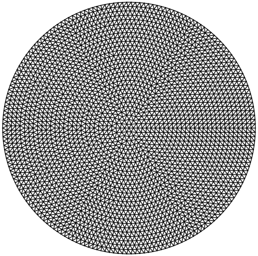
whence $\#\mathcal{T} \approx h^{-2} |\log h|$. Then,

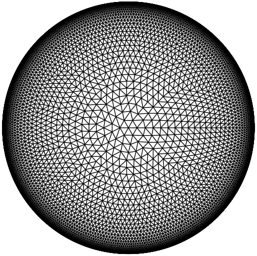
$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \lesssim h |\log h| \, \|f\|_{C^{1-s}(\overline{\Omega})}.$$

Numerical experiment

Exact solution: if $\Omega = B(0,1) \subset \mathbb{R}^2$ and $f = 1$, then $u(x) = C(r^2 - |x|^2)_+^s$.

Value of s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Uniform $\mathcal{T}$	0.497	0.496	0.498	0.500	0.501	0.505	0.504	0.503	0.532
Graded $\mathcal{T}$	1.066	1.040	1.019	1.002	1.066	1.051	0.990	0.985	0.977





Obstacle problem (with R. Nochetto & A. Salgado)

Given two smooth enough functions $f, \chi \colon \Omega \to \mathbb{R}$, find $u \colon \mathbb{R}^n \to \mathbb{R}$, supported in Ω, such that

$$\begin{aligned} u &\geq \chi && \text{in } \Omega, \\ (-\Delta)^s u &\geq f && \text{in } \Omega, \\ (-\Delta)^s u &= f && \text{whenever } u > \chi. \end{aligned}$$

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Can equivalently be written as a variational inequality:

Find $u \in \mathcal{K}$ such that

$$[\![u, u - v]\!] \leq (f, u - v) \quad \forall v \in \mathcal{K},$$

where $\mathcal{K}$ denotes the convex set $\mathcal{K} = \{ v \in \widetilde{H}^s(\Omega) \colon v \geq \chi \text{ a.e. in } \Omega \}$.

Assumptions

- **Domain:** $\partial\Omega$ is Lipschitz, and satisfies an exterior ball condition.

- **Data:** from now on,

$$\chi \in C^{2,1}(\Omega), \qquad 0 \le f \in \mathcal{F}_s(\overline{\Omega}) = \begin{cases} C^{2,1-2s}(\overline{\Omega}), & s \in \left(0, \frac{1}{2}\right) \\ C^{1,2-2s}(\overline{\Omega}), & s \in \left[\frac{1}{2}, 1\right) \end{cases}.$$

- We assume that $\chi < 0$ on $\partial\Omega$, so that
  - the behavior of solutions near $\partial\Omega$ is dictated by an elliptic (linear) problem;
  - the nonlinearity is constrained to the interior of the domain.

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- We assume that $\chi < 0$ on $\partial\Omega$, so that
  - the behavior of solutions near $\partial\Omega$ is dictated by an elliptic (linear) problem;
  - the nonlinearity is constrained to the interior of the domain.

- **Non-locality:** gluing interior and boundary estimates is not straightforward!
  If $\eta \equiv 1$ in a neighborhood of x_0, then it does not follow that

$$(-\Delta)^s(\eta u)(x_0) = (-\Delta)^s u(x_0).$$

Regularity in $\mathbb{R}^n$

Theorem (Caffarelli, Salsa & Silvestre (2008))

For the obstacle problem in $\mathbb{R}^n$, if $f \in \mathcal{F}_s(\mathbb{R}^n)$ and $\chi \in C^{2,1}(\mathbb{R}^n)$, then the solution u belongs to $C^{1,s}(\mathbb{R}^n)$.

(In particular, $u \in H^{1+s-\varepsilon}_{loc}(\mathbb{R}^n)$ for all $\varepsilon > 0$.)

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(In particular, $u \in H^{1+s-\varepsilon}_{loc}(\mathbb{R}^n)$ for all $\varepsilon > 0$.)

Moral: free boundary regularity is not any worse than boundary regularity for the linear problem.

Hope: prove regularity in weighted Sobolev spaces.

Regularity for the obstacle problem on Ω

- **Interior regularity:** Caffarelli-Salsa-Silvestre's theorem + localization argument.
- **Boundary regularity:** use the result for the linear Dirichlet problem.

Regularity for the obstacle problem on Ω

- **Interior regularity:** Caffarelli-Salsa-Silvestre's theorem + localization argument.
- **Boundary regularity:** use the result for the linear Dirichlet problem.

Theorem

Let $u \in \widetilde{H}^s(\Omega)$ be the solution to the fractional obstacle problem. Then, for every $\varepsilon > 0$ we have that $u \in \widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)$ with the estimate

$$\|u\|_{\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)} \leq \frac{C}{\varepsilon},$$

with $C > 0$ depending on $\chi, s, n, \Omega, \|f\|_{\mathcal{F}_s(\overline{\Omega})}$.

Finite element approximation

- **Discrete problem:** find $u_h \in \mathcal{K}_h = \{v_h \in V_h \colon v_h \geq \Pi_h \chi\}$ such that

$$[\![u_h, u_h - v_h]\!] \leq (f, u_h - v_h) \quad \forall v_h \in \mathcal{K}_h.$$

- Weighted Sobolev regularity $\Rightarrow$ **graded meshes.**

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- **Error bound:** writing

$$\|u - u_h\|^2_{\widetilde{H}^s(\Omega)} = [\![u - u_h, u - \Pi_h u]\!] + [\![u - u_h, \Pi_h u - u_h]\!],$$

  we reach

$$\frac{1}{2}\|u - u_h\|^2_{\widetilde{H}^s(\Omega)} \leq \frac{1}{2}\|u - \Pi_h u\|^2_{\widetilde{H}^s(\Omega)} + [\![u - u_h, \Pi_h u - u_h]\!].$$

- **Interpolation error** can be bounded by

$$\|u - \Pi_h u\|_{\widetilde{H}^s(\Omega)} \leq C h^{1-2\varepsilon} \|u\|_{\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)}.$$

Thus,

$$\|u - u_h\|^2_{\widetilde{H}^s(\Omega)} \le Ch^{2(1-2\varepsilon)} \|u\|^2_{\widetilde{H}^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)} + (u - u_h, \Pi_h u - u_h)_s.$$

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- **Second term in RHS:** integrate by parts and use discrete variational inequality,

$$(u - u_h, \Pi_h u - u_h)_s \leq \sum_{T \in \mathcal{T}} \int_T (\Pi_h(u - \chi) - (u - \chi)) \left((-\Delta)^s u - f\right).$$

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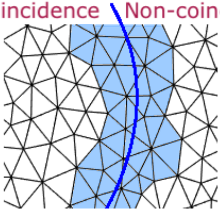
Coincidence Non-coincidence



Using the interior regularity $u \in C^{1,s}(\Omega)$ we deduce:

- $(-\Delta)^s u \in C^{1-s}(\Omega)$,
- $u - \chi \in C^{1,s}(\Omega)$.

So, in these elements we have

$$\left|\left((-\Delta)^s u - f\right)\left(\Pi_h(u-\chi) - (u-\chi)\right)\right| \le Ch^2.$$

Convergence rate

Theorem

$0 \leq f \in \mathcal{F}_s(\overline{\Omega})$ and assume that $\chi \in C^{2,1}(\Omega)$ is such that $\chi < 0$ on $\partial\Omega$. Considering shape-regular graded meshes as before, if h is sufficiently small, then it holds that

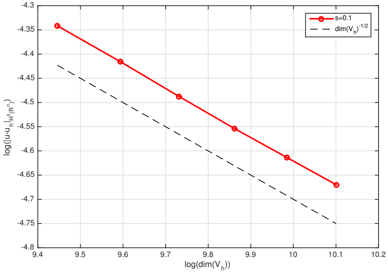
$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \lesssim h|\log h|.$$

Numerical experiments

Problem: let $\Omega = B(0,1) \subset \mathbb{R}^2$, and consider f, χ so that the exact solution is

$$u(x) = (1 - |x|^2)_+^s \, p_2^{(s)}(x),$$

where $p_2^{(s)}$ is a certain Jacobi polynomial of degree two.

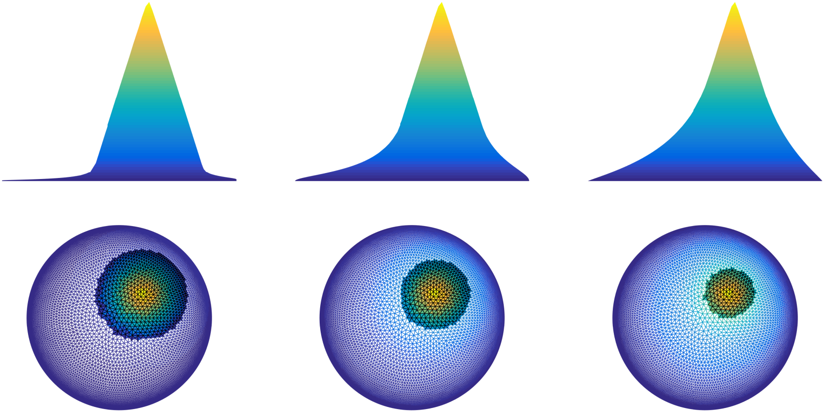


Left: $s = 0.1$; right: $s = 0.9$. The rate observed in both cases is $\approx h$.

Qualitative behavior

Problem: let $\Omega = B(0,1) \subset \mathbb{R}^2$, $f = 0$ and

$$\chi(x) = \frac{1}{2} - |x - x_0|, \quad \text{with } x_0 = (1/4, 1/4).$$



$s = 0.1$ $\qquad$ $s = 0.5$ $\qquad$ $s = 0.9$

Fractional minimal surfaces (preliminary work with R. Nochetto & W. Li)

- **Interaction:** given $s \in (0, 1/2)$ and two disjoint sets $A, B \subset \mathbb{R}^n$, define

$$I(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+2s}} \, dy dx.$$

- **Problem:** suppose we are given $\Omega, \tilde{E} \subset \mathbb{R}^n$ with $\tilde{E} \cap \Omega = \emptyset$. We want to define an extension E of $\tilde{E}$ into Ω so that it minimizes a certain nonlocal perimeter.

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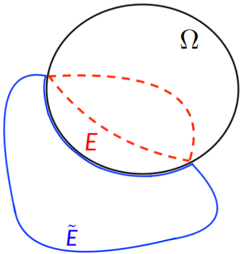
- Minimize $I(E, E^c)$ among all extensions E: take care of interactions
  - between $E \cap \Omega$ and $\mathbb{R}^n \setminus E$,
  - between $\tilde{E}$ and $\Omega \setminus E$.



- **Nonlocal s-perimeter of E in Ω:** (Caffarelli, Roquejoffre & Savin (2010))

$$\mathrm{Per}_s(E,\Omega) := I(E\cap\Omega, \mathbb{R}^n\setminus E) + I(E\setminus\Omega, \Omega\setminus E).$$

- **Minimal sets:** a measurable set $E\subset\mathbb{R}^n$ is s-minimal in Ω if, for every measurable set F such that $E\setminus\Omega = F\setminus\Omega$,

$$\mathrm{Per}_s(E,\Omega) \le \mathrm{Per}_s(F,\Omega).$$

- **Euler-Lagrange equation:** a set E is s-minimal in Ω if and only if

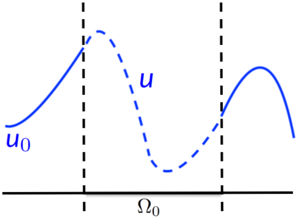
$$(-\Delta)^s\left(\chi_E - \chi_{\mathbb{R}^n\setminus E}\right) = 0, \text{ along } \partial E.$$

Graph minimal surfaces

Assume $\Omega = \Omega_0 \times \mathbb{R}$, and that

$$\tilde{E} = \{x = (x', x_n) \in \mathbb{R}^n : x_n \le u_0(x')\},$$

where $u_0 : \mathbb{R}^{n-1} \setminus \Omega_0 \to \mathbb{R}$ is given.



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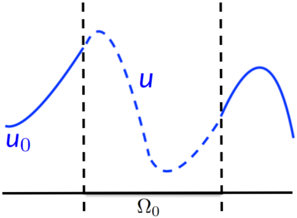


We seek for $u \colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that $u = u_0$ in $\mathbb{R}^n \setminus \Omega_0$, and

$$\int_{\mathbb{R}^{n-1}} g_s\left(\frac{u(y') - u(x')}{|x' - y'|}\right) \frac{u(y') - u(x')}{|x' - y'|^{n-1+2(s+1/2)}} \, dy' = 0 \text{ in } \Omega_0,$$

where

$$g_s(r) = \frac{1}{r}\int_0^r \frac{1}{(1+\rho^2)^{\frac{n+2s}{2}}} \, d\rho.$$

Finding an s-nonlocal minimal surface in $\mathbb{R}^n$ becomes a nonhomogeneous problem for a **nonlinear, degenerate diffusion operator of order** $s + \frac{1}{2}$ in $\mathbb{R}^{n-1}$.

Discretization

- **Finite element space:** let

$$\mathbb{V}(\mathcal{T}) = \{v_h \in C^0(\overline{\Omega_0}) \colon v_h\big|_K \in \mathcal{P}_1 \; \forall K \in \mathcal{T}\}.$$

- **Discrete problem:** find $u_h \in \mathbb{V}(\mathcal{T})$ such that $u_h = \Pi_h u_0$ in $\mathbb{R}^{n-1} \setminus \Omega_0$ and, for all $v_h \in \mathbb{V}(\mathcal{T})$,

$$\iint g_s\left(\frac{u_h(y') - u_h(x')}{|x'-y'|}\right) \frac{(u_h(y') - u_h(x'))(v_h(y') - v_h(x'))}{|x'-y'|^{n+2s}}\, dy' = 0.$$

- **L^2-gradient flow:** initial guess $u_h^0 \in \mathbb{V}(\mathcal{T})$ and time step τ. Given $u_h^k \in \mathbb{V}(\mathcal{T})$, find $u_h^{k+1} \in \mathbb{V}(\mathcal{T})$ such that

$$\frac{1}{\tau}\left(u_h^{k+1} - u_h^k, \varphi_i\right) = \iint g_s\left(\frac{u_h^k(y') - u_h^k(x')}{|x'-y'|}\right) \frac{(u_h^k(y') - u_h^k(x'))(\varphi_i(y') - \varphi_i(x'))}{|x'-y'|^{n+2s}}\, dy',$$

$$\forall 1 \le i \le \mathcal{N}.$$

Energy

The solution u minimizes the energy

$$I_s[u] = \iint_{(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\setminus(\Omega_0^c\times\Omega_0^c)} G_s\left(\frac{u(x)-u(y)}{|x-y|}\right)\frac{1}{|x-y|^{n-2+2s}}\,dy\,dx,$$

where G_s is defined as

$$G_s(a) := \int_0^a \frac{a-\rho}{(1+\rho^2)^{\frac{n+2s}{2}}}d\rho \qquad (G_s' = g_s).$$

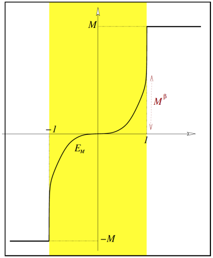
Since $a \le C(G_s(a)+1)$, we have

$$|u|_{W^{1,2s}(\Omega_0)} \le C\, I_s[u] + C(\Omega_0).$$

Convergence

- **Open question:** how regular are nonlocal minimal surfaces?
- **Stickiness phenomenon:** boundary datum may not be attained continuously!

(Dipierro, Savin & Valdinoci (2017))



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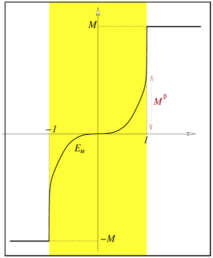

Theorem (energy consistency)

If $u \in W_1^{2t}(\Omega_0)$ for some $t > s$, then $\lim_{h\to 0} I_s[u_h] = I_s[u]$.

Theorem (convergence)

If we have energy consistency, then

$$\lim_{h\to 0} \|u - u_h\|_{W_1^{2s'}(\Omega_0)} = 0, \quad \forall s' \in [0, s).$$

Experiments

Problem: $\Omega = B(0,1)$, $u_0 = \chi_{B(0,3/2)}$ and $s = 0.25$.

Experiments

Problem: $\Omega = B(0,1) \setminus \overline{B(0,1/2)}$, $u_0 = \chi_{B(0,1/2)}$ and $s = 0.25$.

Concluding remarks

- **Fractional Laplacian** $(-\Delta)^s$**:** nonlocal operator of order $0 < 2s < 2$. Computational challenges include dealing with non-integrable singularities and unbounded domains.
- **Boundary behavior:** solutions of the problems discussed behave as $\operatorname{dist}(x, \partial\Omega)^s$ $\Rightarrow$ characterize regularity in weighted Sobolev spaces $\Rightarrow$ use graded meshes.
- **Fractional obstacle problem:** behavior near the free boundary may not be any worse than behavior near $\partial\Omega$.
- **Minimal surfaces:** leads to nonlinear, degenerate diffusion problem. Solutions may exhibit discontinuities near $\partial\Omega$.

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Thank you!